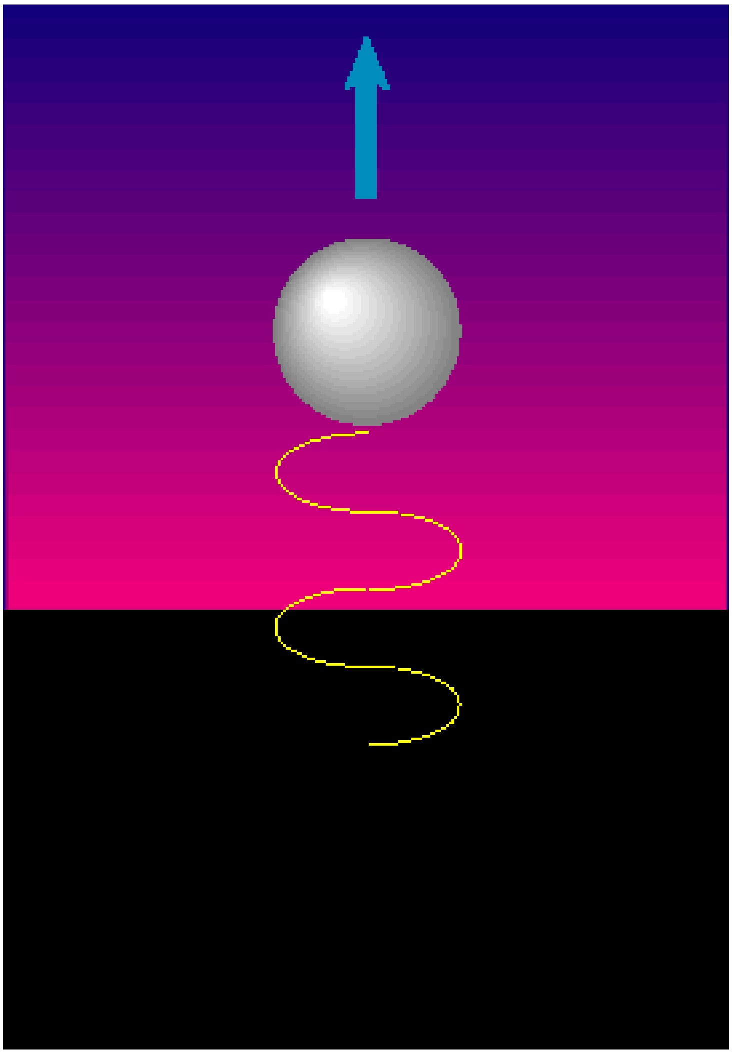


The non-relativistic velocity of an uncharged object emitting a photon

Imagine an uncharged object of mass m_o moving linearly through the vacuum of space with low initial velocity v_o. A moment later, the object emits a photon of momentum p from its surface. The object recoils in the opposite direction to the photon with a new velocity of $v_1=v_o+dv$. Assuming the mass of the object remains the same as before, what is the velocity v_1 after emission of the photon?

According to the principle of conservation of linear momentum, which states that ‘when no resultant external force [such as gravity or air friction] acts on a system, the total momentum of the system remains constant in magnitude and direction'[1], the problem can be mathematically stated as:

$$m_o v_o + p = m_o v_1$$

or

$$m_o v_o + p = m_o(v_o + dv)$$

Rearranging terms gives,

$$dv = \frac{p}{m_o}$$

Formally integrating the differential term dv,

$$\int dv = \frac{p}{m_o}$$

$$v + C = \frac{p}{m_o}$$

Prior to the discrete emission of the photon, the object had a velocity equal to its initial velocity. Therefore $C = -v_o$, and so the non-relativistic recoil velocity for the object emitting just one photon is given by:

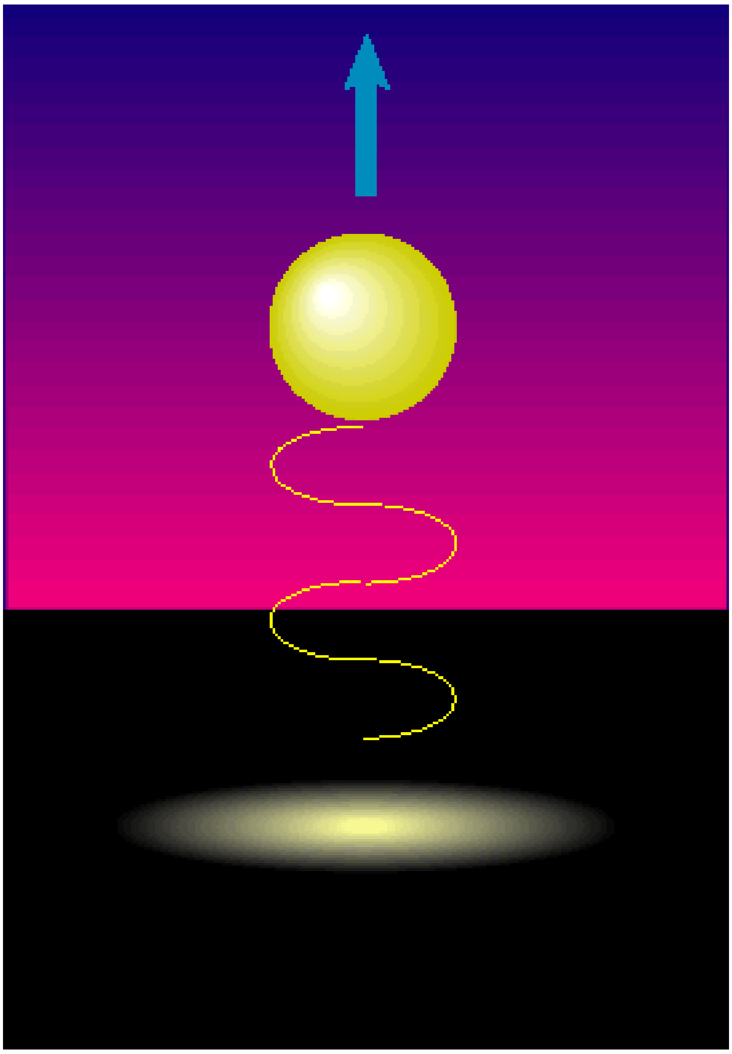
$$v_1 = v_o + \frac{p}{m_o}$$

EXAMPLE

Suppose a sodium atom of mass $m_o = 3.8x10^{-26}$ kilogram, initially at rest, emits a photon of wavelength $\lambda = 5.89x10^{-7}$ metre. The recoil velocity of the atom is as follows:

$$v_1 = \frac{p}{m_o} = \frac{h/\lambda}{m_o} = \frac{6.6260755x10^{-34} J.s/5.89x\,10^{-7} m}{3.8x10^{-26} kg} = 0.029 \text{ metres per second.}$$

[1] Zemansky et al 1982, p.147.



The non-relativistic velocity of a charged object emitting a photon

But what happens when the object is electrically charged? Instead of an uncharged object emitting just one photon at a time as it moves through space, a charged object accelerating through space is said to be emitting electromagnetic radiation continuously. And since the charge will increase speed after the emission of the radiation, the frequency and strength of the electromagnetic field in the radiation emitted by the charge will also increase. This subtle fact will undoubtably change the above equation for the uncharged case drastically.

In the words of Dr David J. Griffiths of the Department of Physics at Reed College, USA, who has studied the problem:

‘According to the laws of classical electrodynamics, an accelerating charge radiates [electromagnetic energy]....The radiation evidently exerts a force back on the charge - a recoil force rather like that of a bullet on a gun...’[2]

This force is called the *radiation reaction force*. Continuing with the quote:

‘The Abraham-Lorentz formula for [this] radiation reaction force...consistent with conservation of energy [is],

$$\boldsymbol{F} = \frac{1}{4\pi\varepsilon_o}.\frac{2}{3}.\frac{Q^2}{c^3}.\frac{d\boldsymbol{a}}{dt}$$

‘...the Abraham-Lorentz formula has disturbing implications, which are not entirely understood 80 years after the law was first introduced.’[3]

The reason for making this remarkable statement can be seen by looking at the formula for acceleration. If the charged object has mass m, then according to Newton’s second law of motion:

$$m\frac{dv}{dt} = \frac{1}{6\pi\varepsilon_o}.\frac{Q^2}{c^3}.\frac{d^2v}{dt^2}$$

or

$$\frac{d^2v}{dt^2} - \frac{6\pi\varepsilon_o mc^3}{Q^2}.\frac{dv}{dt} = 0$$

After solving this differential equation of the second order, we obtain:

$$v = a_o\kappa\left(e^{t/\kappa} - 1\right), \quad \text{where } \kappa = \frac{Q^2}{6\pi\varepsilon_o mc^3}$$

This is the non-relativistic recoil velocity of a charged object of mass m, charge Q and initial acceleration a_o. The acceleration of the charged object is, therefore:

$$a = \frac{dv}{dt} = a_o e^{t/\kappa}$$

2 Griffiths 1989, pp.432-439.

3 Dr Nunzio Tralli, former professor of physics at Long Island University, USA, gives full derivation of the Abraham-Lorentz formula in his 1963 book, Classical Electromagnetic Theory, pp.271-276.

This equation implies a spontaneous exponential increase in the object's acceleration over time t, even when no external force is applied.

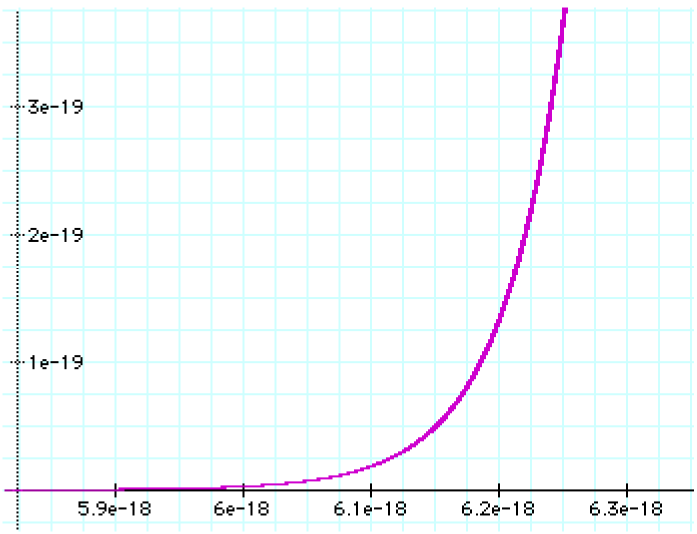
Is this possible in reality?

EXAMPLE

Suppose an electron of mass $m_o = 9.10939x10^{-31}$ kilogram and electric charge $Q = 1.6021892x10^{-19}$ C, has an initial acceleration of 0.1 $m.s^{-2}$ as it emits a photon of a particular wavelength. The recoil velocity of the electron after t = 6.2×10^{-18} second is as follows:

$$v = (0.1)\kappa\left(e^{t/\kappa} - 1\right)$$

where $$\kappa = \frac{\left(1.6021892x10^{-19}C\right)^2}{6\pi\left(8.8541853x10^{-12}C^2N^{-1}m^{-2}\right)\left(9.10939x10^{-31}kg\right)\left(3x10^8 m.s^{-1}\right)^3}$$



Or roughly $v = 1.37 \times 10^{-19}$ $m.s^{-1}$.

Slow at first. But give it a little more time while maintaining the charge and the frequency of the radiation and the velocity will quickly approach the speed of light.